Theorems A′, B′, C′, p. 80]. It is also worth noting that in the Law of appearance of p, if $p = 5m \pm 1$, then $A^{p-1} \equiv I \pmod{p}$ (solving [**1**, Problem 69, p. 33]), and if $p = 5m \pm 2$, then $A^{p+1} \equiv -I \pmod{p}$.

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Geometric Proofs of the Weitzenböck and Hadwiger-Finsler Inequalities

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Problem 2 on the Third International Mathematical Olympiad in 1961 read [**3**]:

Let a, b, c be the sides of a triangle, and T its area. Prove:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,T. \tag{1}$$

In what case does equality hold?

This inequality is well known in the literature [**1, 4, 8**] as *Weitzenböck's Inequality* (sometimes spelled Weizenbock) from a paper published in 1919 by R. Weitzenböck in *Mathematische Zeitschrift*. Many analytical proofs of the inequality are known—see the above references. The "official" solution to the Olympiad problem appears to have been trigonometric, employing the identities $T = bc \sin A$, $a^2 = b^2 + c^2 - 2bc \cos A$, and $(\sqrt{3} \sin A + \cos A)/2 = \cos(A - 60^\circ)$ [**3**]. Note that A denotes the vertex opposite the side of length a, etc.

There is a very nice geometrical interpretation of this inequality that seems to have been overlooked. If one multiplies both sides of inequality (1) by $\sqrt{3}/4$, then Weitzenböck's inequality can be written as

$$T_a + T_b + T_c \geq 3T, \tag{2}$$

where T_s denotes the area of an equilateral triangle with side length s. The situation is illustrated in FIGURE 1, where (2) states that the sum of the areas T_a, T_b, and T_c of the three shaded equilateral triangles is at least three times the area T of the white triangle.

We now present a purely geometric proof of Weitzenböck's inequality in the form given by (2). Since the proof uses the Fermat point of the original triangle, we first discuss this point. The *Fermat point* of a triangle ABC is the point F in or on the triangle for which the sum $AF + BF + CF$ is a minimum (this is also known as the solution

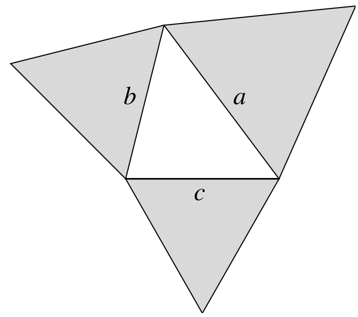

Figure 1

to Steiner's Problem). See FIGURE 2(a). When each of the angles of the triangle is smaller than 120°, the point F is the point of intersection of the lines connecting the vertices A, B, and C to the vertices of equilateral triangles constructed outwardly on the sides of the triangle, as shown in FIGURE 2(b). Furthermore, each of the six angles at F measures 60°. When one of the vertices of triangle ABC measures 120° or more, then that vertex is the Fermat point. For a variety of proofs of these rather remarkable results, see [**2, 6, 7**].

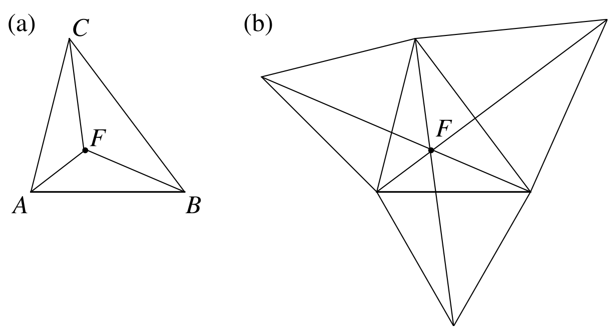

Figure 2

We are now in a position to prove (2). We first consider the case where each angle of the triangle is less than 120°. Let x, y, and z denote the lengths of the line segments joining the Fermat point F to the vertices, as illustrated in FIGURE 3(a), and note that the two acute angles in each triangle with a vertex at F sum to 60°. Hence the equilateral triangle with area T_c is the union of three triangles congruent to the dark

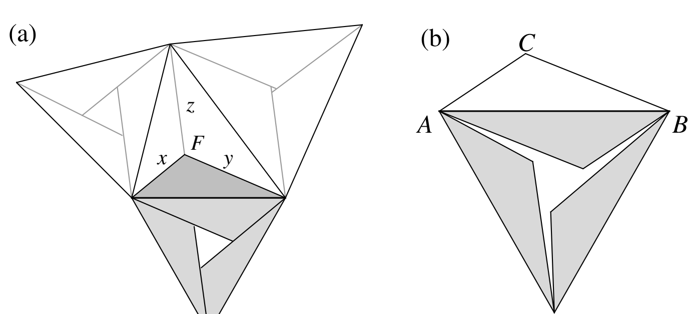

Figure 3

gray shaded triangle with side lengths x, y, and c, and an equilateral triangle with side length $|x - y|$. The same is true of the other triangles sharing the vertex F, and hence

$$T_a + T_b + T_c = 3T + T_{|x-y|} + T_{|y-z|} + T_{|z-x|}, \tag{3}$$

which establishes (2) in this case since $T_{|x-y|}$, $T_{|y-z|}$, and $T_{|z-x|}$ are each nonnegative.

When one angle (say C) measures 120° or more, then, as illustrated in FIGURE 3(b), we have

$$T_a + T_b + T_c \geq T_c \geq 3T \tag{4}$$

which completes the proof.

It follows from (3) that we have equality in (1) and (2) if and only if $x = y = z$, so that the three triangles with a common vertex at F are congruent and hence $a = b = c$, i.e., the original triangle is equilateral.

The relationship in (3) is actually stronger than the Weitzenböck inequality (1), and enables us to now prove another inequality, itself stronger than (1), the *Hadwider-Finsler Inequality* [**1, 8**]: If a, b, and c are the lengths of the sides of a triangle with area T, then

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,T + (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{5}$$

In terms of areas of triangles, (5) is equivalent to

$$T_a + T_b + T_c \geq 3T + T_{|a-b|} + T_{|b-c|} + T_{|c-a|}. \tag{6}$$

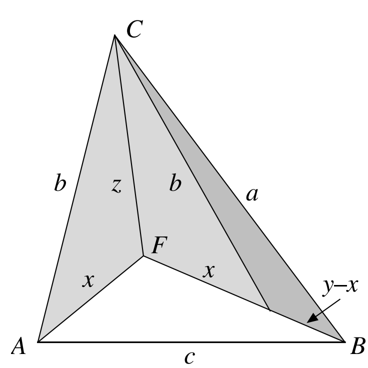

Figure 4

To prove that (3) implies (6) when all three angles measure less than 120°, we need only show that $|x - y| \geq |a - b|$, $|y - z| \geq |b - c|$, and $|z - x| \geq |c - a|$. Without loss of generality, assume that $a \geq b \geq c$. Within triangle ABC reflect the triangle with sides of length b, x, and z about the segment of length z as shown in FIGURE 4, to create two congruent light gray triangles (recall that each of the three angles at F measures 120°). Then in the dark gray triangle we have $b + y - x \geq a$, or equivalently, $y - x \geq a - b$. The other two inequalities are established similarly, and hence from (3) we have

$$\begin{aligned} T_a + T_b + T_c &= 3T + T_{|x-y|} + T_{|y-z|} + T_{|z-x|} \\ &\geq 3T + T_{|a-b|} + T_{|b-c|} + T_{|c-a|}. \end{aligned}$$

In the case where one angle (say C) measures 120° or more, we have $z = 0$, $x = b$, and $y = a$. We refine the inequality in (4) to $T_a + T_b + T_c \geq 3T + T_{|a-b|} + T_a + T_b$ (see FIGURE 3(b)), and note that $a \geq |b - c|$, and $b \geq |c - a|$, from which (6) follows.

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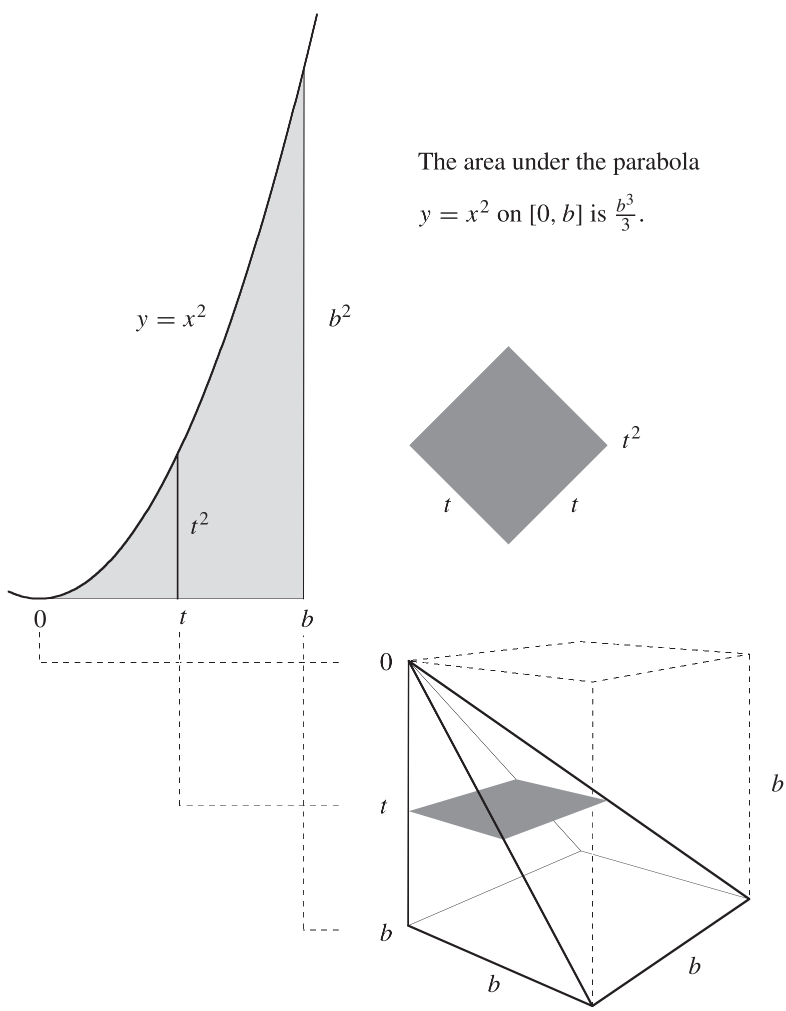
Proof Without Words: Area of a Parabolic Segment

The area under the parabola

$y = x^2$ on $[0, b]$ is $\frac{b^3}{3}$.



$$\int_0^b x^2 dx = \text{Volume of Pyramid} = \frac{1}{3} \cdot \text{height} \cdot \text{base} = \frac{1}{3} \cdot b \cdot b^2 = \frac{b^3}{3}$$

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